Homework 3

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Homework 3 (due Wed. Feb 16)

1. Suppose you have n independent samples $x_1, ..., x_n$ of a random variable η; show that if $m = (1/n)\sum_{i=1}^n x_i$, then $(1/n)\sum(x_i - m)^2$ is not an unbiased estimate of the variance of η, while $(1/(n-1))\sum(x_i - m)^2$ is an unbiased estimate. Suggestion: To see what is going on, try first the case $n = 2$. Note: these calculations are independent of any assumed form for the density.

2. Consider a vector valued Gaussian random variable ξ_1, ξ_2, with pdf

$$f(x_1, x_2) = f(\mathbf{x}) = \frac{\alpha}{\mathbf{2}\pi} \exp(-(\mathbf{x} - \mathbf{m}, \mathbf{A}(\mathbf{x} - \mathbf{m}))/\mathbf{2}), \tag{1}$$

where A is a symmetric positive definite matrix. Show that $\alpha = \sqrt{\det A}$ and $A = C^{-1}$, where C is the covariance matrix.

3. Let $(\Omega, \mathcal{B}, P)$ be a probability space, A an event with $P(A) > 0$, and $P_A(B) = P(B|A)$ for every event B in $\mathcal{B}$. Show that $(\Omega, \mathcal{B}, P_A)$ satisfies all the axioms for a probability space.

4. let η_1, η_2 be two random variables with joint pdf $Z^{-1}\exp(-x_1^2 - x_2^2 - x_1^2 x_2^2)$, where Z is a normalization constant. Evaluate $E[\eta_1 \eta_2^2 | \eta_1]$.

5. Let η be the number that comes up when you throw a die. Evaluate $E[\eta|(\eta - 3)^2]$ (you may want to present it as a table of its values for different values of η).

6. Suppose η is a random variable such that $\eta = 0$ with probability p and $\eta = 1$ with probability $1 - p$. Suppose your prior distribution of p is $P(p = 1/2) = 0.5$ and $P(p = 3/4) = 0.5$. Now you make an experiment and find $\eta = 1$. What is the posterior distribution of p? Suppose you make another, independent, experiment, and find again $\eta = 1$. What happens to the posterior distribution? Suppose you keep on making experiments and keep on finding $\eta = 1$. What happens to the posterior distributions? Why does this make sense?